DERIVING AREA FORMULA FOR RECTANGLES WITH OPEN-ENDED QUESTIONS

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RATIONALE

As mentioned in the title of this module, this strand of ‘area’ is approached through open-ended questions, necessitated by the following rationale.

When students learn about the area of rectangles, if teachers can involve students in the development of its formula, their understanding can be elevated to a generic level concerning the basic concepts. Such understanding would further widen their perspective in solving problems, with the formula or otherwise. On the contrary, without such enriched understanding, solving problems of area would become merely procedure driven, which may deprive students of the interesting experience of meaningful mathematics that makes sense, and the concomitant growth of lifelong exploration. With open-ended questions, it is more likely that non-Chinese speaking (NCS) students would no longer find lessons boring and exercise requirements inaccessible.

Open-ended questions, such as (i) ‘how’, (ii) ‘why’ and (iii) ‘what’ with multiple responses, invite students to express, verbally or otherwise, their ideas and thinking process. This type of questions is essential for students’ mathematical development since it focuses on conceptual understanding, rather than memorization of formula and procedure. Open-ended responses provide a valuable reflective tool to teachers to become aware of what sense students have made from their teaching, whether right or wrong, complete or inadequate, etc., pointing at timely remedial actions if necessary.

In NCS classrooms, teachers are often worried that open-ended questions are too challenging for ethnic minority (EM) students to answer, and prefer to ask simpler, procedural questions with unique and standard answers. In such a simplistic way, many EM students can be alienated from active thinking. Then teachers would think that they are unmotivated and off-task during lessons. Consequently, teachers would become further unconfident to involve EM students in any classroom conversation. This forms a vicious circle.

Signs of reversing such a vicious circle into a virtuous one is visible when a project school experimented with some of the suggestions in this module. They have embarked on open-ended questions and responses, where students’ eagerness was maintained and enhanced by well scaffolded strategies, and their capacity to learn through open-ended queries was ascertained. The students’ consequential interest and willingness to engage in activities and conversation made teachers more confident in implementing similar strategies in subsequent lessons. This may be the beginning of a virtuous circle. The above experience is summarized and developed into the targets, strategies and materials shown in sections below.

LEARNING TARGETS

In the lessons, students are expected to
1. investigate the formula for areas of rectangles,
2. enhance their communication skills, and
3. develop their exploratory spirit and related skills.

Target 1 is a subject skill. Target 2 consists of sharing understanding, clarifying misperceptions, evaluating ideas, etc. Target 3 relates to discovery, without fear of uncertainty or making mistakes. All these skills are interdependent, mutually reinforcing and embedded in each other. Therefore, they are developed simultaneously in the same learning process, hence the strategies in the section below, featuring open-ended questions, hands-on explorations and problems created by students, etc.

LEARNING AND TEACHING STRATEGIES

For EM students, whether it is CMI or EMI, both are foreign to them. Unfortunately, open-ended questions and responses indeed rely more on language usage than simplistic Q&A. But avoidance of richer language in the learning process runs the risk of failure to reach a deep understanding. Therefore, the following strategies are suggested.

1. **Supporting students with tools:**
   Teachers can support students' comprehension of an open-ended question by presenting it with tools, such as rulers, grid paper, tiles or other manipulatives or visuals. Tools help students approach the question in two ways, (i) students can better comprehend the question and (ii) tools serve as thinking hints that trigger their prerequisite knowledge and skills towards the solution.

2. **Following an open-ended question with multiple-choice sub-questions:**
   If students have no idea of handling an open-ended question, teachers can let them choose from multiple-choice sub-questions, through which student engagement can begin with, at least, retrieval of their relevant prior knowledge and skills, ready for application. Open-ended questions can then become a natural follow-up, e.g. asking students to explain why they took or rejected certain choices.

3. **Elaborating on student responses:**
   Teachers may repeat, rephrase or extend students' responses to more precise daily language, to a more standard mathematical term or to an alternative or better line of reasoning, etc. More important is that students' contributions are retained, recognized and appreciated, although they may not have been expressed in the best way. This paves the way towards acquiring the generic skills of enriched communication.

4. **Gradating problem-solving exercises:**
   Problem-solving exercises can be designed as a connected series of open-ended questions where the level of difficulty is gradually raised.

5. **Encouraging students to create their own problems:**
   Students creating a problem for classmates to solve often prompts and leads to a deeper understanding than solving a given problem. It can start with modification of a problem they have solved. Group work helps.

The above 5 strategies gravitate onto these two themes:
(i) Beyond algorithm: developing conceptual understanding as well as procedural knowledge of arithmetic operations.
(ii) Exercise counts: illustrative examples and intelligent practice.

TEACHING NOTES, TASKS, EXERCISES AND MATERIALS

----- for the investigation of area formula of rectangles

A. GENERAL REMINDERS TO TEACHERS

- how to make use of open-ended questions for exploratory learning

For the four points below, readers may simply read their topics first, then head towards the tasks in section B, and come back for the details afterwards.

1. **Providing plentiful time for exploration**
When introducing tasks with open-ended questions, enough time should be provided for students to explore for as many answers as possible. Teachers may walk around to observe how students work, or, in case of group work, how they discuss the matter, clarifying if necessary. They should also welcome and answer students' questions. For students who find it difficult to start their work, teachers may break the overall question into simpler and smaller ones.

2. **Introducing 'turn and talk'[1]**
After allowing a reasonable time for everyone to have formed some ideas or a complete approach to the task, a few minutes of 'turn and talk' may help develop both the subject and communication skills. 'Turn and talk' means finding a peer (from another group if it was group work) to explain one's idea or approach, and listening to the other's afterwards. Much can be learnt through such talk, especially when different ideas are compared and discussed.

3. **Selecting students' work for sharing**
When observing how students work, teachers can, in their minds, select a few of them for a sharing session to the class later. Teachers should be aware of any different approaches among the students, value all of them and select accordingly. Approaches found in another class but not this one may be brought up and discussed at an appropriate time.

   Do not select, for sharing, only students that come up with correct answers. Wrong or unorganized answers are useful, in that they, calling for elaboration and clarification, can trigger discussion and stimulate critical thinking, leading to deeper understanding.

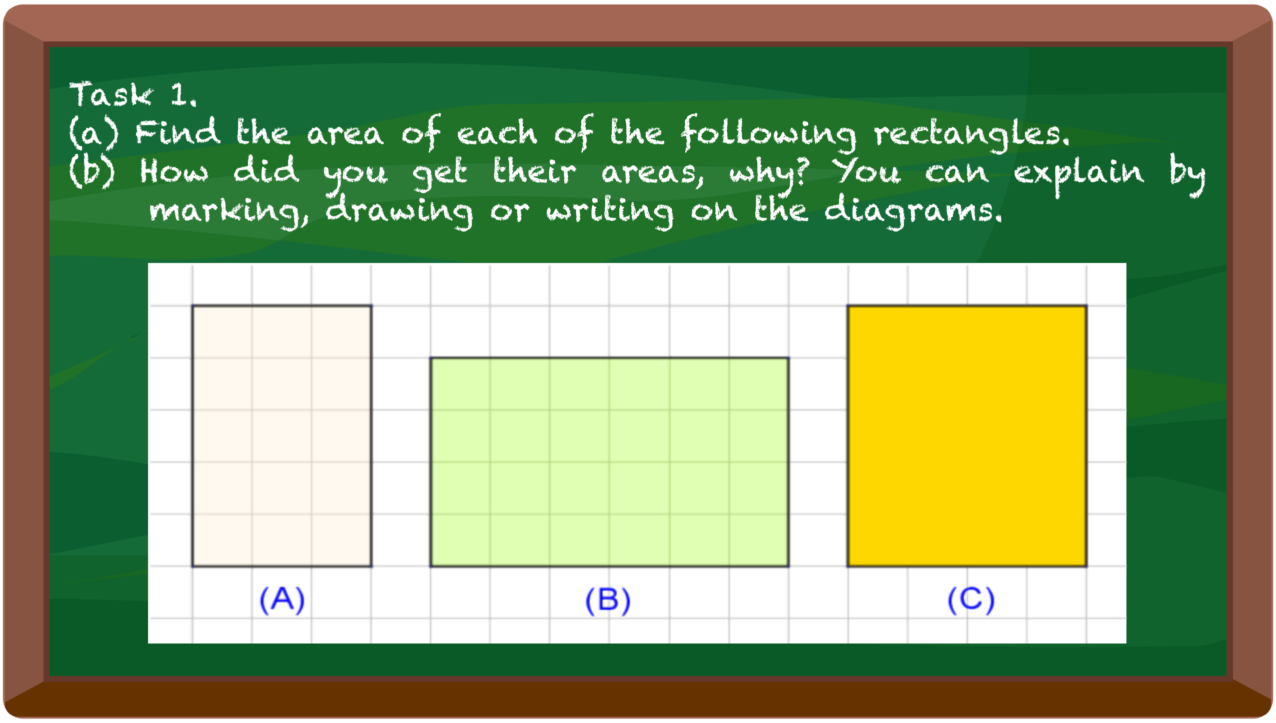
4. **Planning the order of sharing**
Furthermore, it may be advisable to plan the various selected approaches for sharing in a proper sequence. Students work can be arranged in an order that foster discussion through which the learning targets can be better achieved for all, e.g. never placing at the beginning what the teacher deems the standard approach.

1. The idea of 'turn and talk' originates from the article "Discourse: simple moves that work" by Molly Rothermel Rawding & Theresa Wills (2019).

B. SUGGESTED TASKS

There are four tasks suggested in this module. Teachers may assign these tasks to students for pair or group work. Some can also be assigned as exploratory exercises. Printable worksheets for these tasks are provided in the Resource page.

Level 1 Let students explore various methods to find areas of rectangles.



<Examples of probable dialogue between teacher (T) and students (S)>

(i) T: Your answer is right. How did you get it?
S: I counted the number of squares.
T: Good. (T proceeds to show a big rectangle formed by many many 1cm × 1cm tiles.) Look at this rectangle. What's the area? (After 1 minute, whether any answers has come out or not, T carries on.) Thank you for counting so hard. Can you think of a faster way?
………….

(ii) T: How did you get it?
S: Base times height.
T: Why is it base times height? Up to this point, teachers can lead students to work on Task 3.
S: It's the formula.
T: OK, you're right. Can anyone tell us why this formula works?
………….

<Let students proceed to tasks of a higher level>

As shown, dialogues (i) and (ii) above do not stop there. What can be done next? Tasks 2 and 3 in levels 2 and 3 below are recommended extensions for further discussions. These extensions are valuable in that, instead of simply stating or establishing the formula base × height or length × width, students are allowed to gradually form the pre-cursor ideas in their minds.

Level 2 Let students explore various possible rectangles with the same area.

Task 2.
(a) On the 1-cm grid paper draw all possible rectangles with an area of 12 cm².
(b) How do you know that the area is 12 cm²?

<Utilization of virtual manipulatives for question 2(a)>

‘Graph Color Bars’ provides virtual manipulatives suitable for the tasks suggested in this module.
It can be found at https://toytheater.com/graph-color-bars/.
Diagram X shows the page they have designed to introduce the bars, which can be dragged to the empty space to form rectangles. Each bar can be used multiple times.
Students can plan their rectangles with the virtual manipulatives in ‘Graph Color Bars’. If a student chooses certain bars other than the smallest squares to form her rectangles (see Diagram Y), she might discover the area formula sooner.

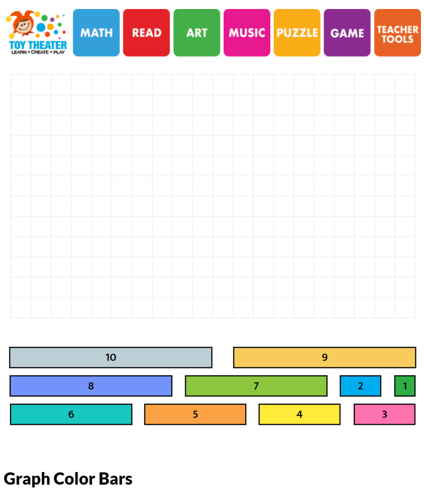


Diagram X

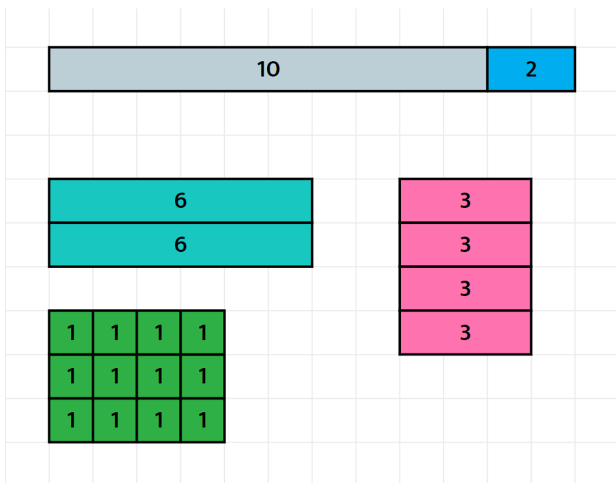


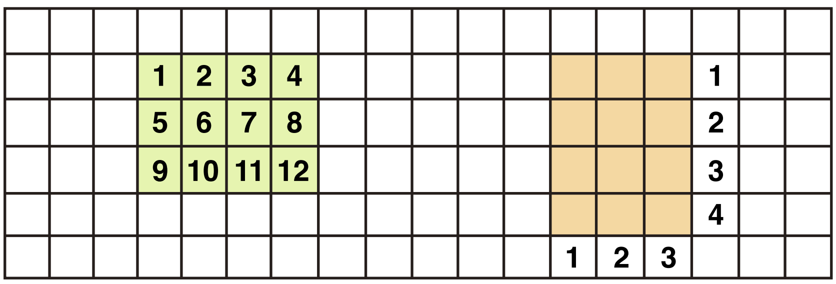
Diagram Y

<Probable student responses to question 2(b) and teacher’s facilitation for learning>

See the diagram below as examples.

(i) Reason A is the most likely authentic answer for 2(b), as they have well learnt to find areas by counting the number of 1 cm × 1 cm tiles. Then T may refer to dialogue (i) in Task 1 above for a proper follow-up.

Reason A Reason B



(ii) If any student manages to come up with Reason B, further elaboration may be invited from the class, to check if they could relate the reason to the concepts of multiplication[2]. For example, as a first step, the number of tiles in one row/column can be identified.

(iii) Another likely answer for 2(b) would be the formula base × height, although rare among EM students. Dialogue (ii) in Task 1 above may provide a hint for follow-up.

(iv) Students should be free to use the shorter or longer side as the base. They can be deemed the same rectangle or two different ones. The teacher needs not point out that they are the same rectangle and hence only one rectangle has been identified. Instead, it would be far better to pose open-ended questions to let students discover that
   (a) either side can form the base (or height) and
   (b) despite the same area, a rectangle can be rotated by 90o or any other angle to make seemingly another rectangle.

<Let students proceed to tasks of a higher level>

This task is designed to prepare students in creating any rectangle of a required area with grids or, further, bars. Creation of those rectangles would trigger students to think of which are the key factors of the area of a rectangle. The next task, Task 3, would lead students to formulate the algorithm to calculate the area of any rectangle.

Level 3 Let students derive the area formula for a rectangle.

Task 3.
(a) Draw all possible rectangles with an area of 18 cm² and label their bases and heights.

(Teacher may provide students with tools for visualization and exploration. Possible choices are 'Graph Color Bars', rulers or a simplified ruler in the form of a paper strip of only 1cm × 1 cm tiles on it, etc.)

2. Interactive simulations from file:///C:/Users/Queenie/Downloads/arithmetic_en.html is recommended for the revision of concepts of multiplication.

(b) When you draw the rectangles, how do you know that all their sizes are 18 cm^2?

It is very likely that students would give their reason by counting the number of unit squares within each rectangle, like this.

How can students be led to the formula “area = base × height”?

A simplistic approach would be just to announce it at this moment. But more generic skills would be learnt if the step by step exploration and discovery on their own in parts (c) and (d) are followed. Several concept developments involved are

– possible improvement from counting each and every unit square, if the total number is many more than 18
– realizing that the number of unit squares along any side of a rectangle is the length of that side
– labelling the length of two adjacent sides
– identifying them as base and height
– generalizing a uniform pattern of finding the areas (not by counting) of rectangles of different shapes and orientations
– extrapolating the same pattern to rectangles whose areas are not 18 cm^2.

(c) For all the rectangles you have drawn, what bases and heights would you suggest? Give whole numbers for your answers, by filling in the table below

	Base	Height	Area
Rectangle A			18 cm^2
Rectangle B			
Rectangle C			

It is not uncommon that some students may give a mixture of right and wrong answers like this:

	Base	Height	Area
Rectangle A	3 cm	6 cm	18 cm^2
Rectangle B	4 cm	6 cm	18 cm^2
Rectangle C			

Then the teacher, instead of just pointing out the error, may lead them on a journey of self-evaluation by, e.g. asking them

– to compare the number pairs for rectangles A and B,
– to observe which rectangle is bigger in their drawings and/or
– to relate the above comparison with the observation,
– etc.

(d) What relation do you find among the base, height and area of each rectangle? Why is there such a relation?

While students can normally give the answer for the first of these two questions, the second one is trickier. The teacher may elicit from students the following ideas through multi-way dialogue:

(i) visualizing cutting each of their rectangles into a number of horizontal bars of the same size (instead of the small 1 cm^2 squares with which they are more familiar),

(ii) where each bar is 1 cm high, and hence the area of the bar can be determined by thinking about how many 1 cm^2 squares can be put in it (that number would happen to be the same as the value of the base),

(iii) counting the number of such bars in the whole rectangle (that number would happen to be the same as the value of the height), and finally

(iv) finding the area of the whole rectangle by relating the area of each bar (i.e. base) and the number of bars (i.e. height).

(v) Even after the successful elicitation of the concepts in (i) – (iv) above, a further idea can be added: going through (i) – (iv) again by changing the horizontal bars in (i) into vertical ones.

Level 4 Let students create an open-ended problem for classmates to solve.[3]

Task 4.

(Teacher prepares various rectangles with whole numbers as their bases and heights, like the following, and give one of them to each student.)

(a) Fill in the blanks.

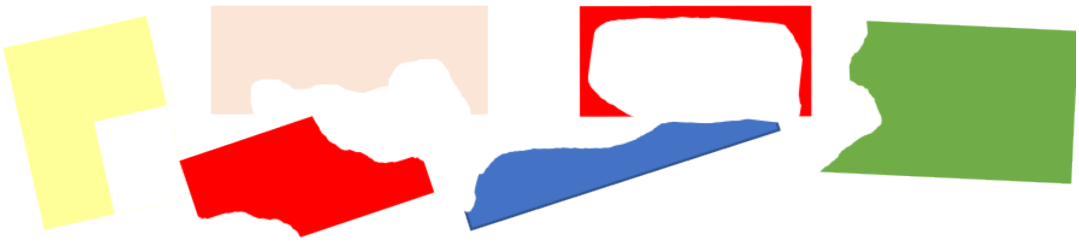
My rectangle:

Base	Height	Area

(b) (i) Tear out part of your rectangle and keep it as a secret to yourself.

(ii) Exchange the remaining (open) part with a classmate. Later, you two will be required to guess the area of each other's original rectangle.

<Possible examples of torn rectangles, the remaining or open part of them>



(c) (i) What is the area of your classmate's original rectangle?

Area of classmate's original rectangle:

(ii) How did you find out that area? Try explaining in detail. (Think about recovering the whole rectangle from the torn one. You may show your ideas by drawing, cutting or measuring, etc.)

3. Idea of this task originates form the problem, "Torn Shapes", retrieved from https://nrich.maths.org/4963.

<Notes to teachers>

(i) There can be a preparatory step for all students or only those who do not have a clue, e.g.

- you can provide them with a blank paper from which to cut out his/her classmate's original rectangle based on the remaining (open) part,
- you can let them try another torn rectangle on 1-cm grid paper (Diagram A) first, before handling the torn rectangle s/he has got from his/her classmate, or
- you can give him/her a transparent 1-cm grid paper, which s/he may superimpose on the torn rectangle.

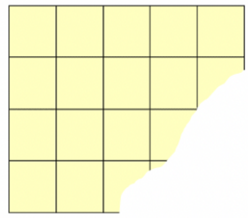


Diagram A

(ii) Students may not follow your instruction strictly and tear two parts away. This is not a bad thing. Although some students may find it impossible to recover the whole rectangle from the torn one such as Diagram B, you should still encourage them to

Diagram B

(iii) In case torn rectangles like those in Diagram C come up, then it would be possible to find a unique answer. But such rectangles must not be dismissed or neglected. The following questions can be asked to bring students to a wider horizon of open-endedness and a more exciting adventure.

Diagram C

- **Is it impossible to find the original rectangle?**
- **Can you identify one possible original rectangle?**
- **Can you identify some more?**
- **What is the smallest area the original rectangle could have had?**
- **Draw a larger possible original rectangle.**
- **What is the largest area it could have had?**

(iv) Torn rectangles like those in diagrams B and C may never appear in your class when they attempt Task 4. Even if that is the case, you may have thought of bringing them up as a grand finale of this module. Indeed, that would go a long way in meeting the targets of concept understanding, communication and exploration rather well in terms of width and depth.

ASSESSMENT TOOLS

Assessment *for* learning has been embedded in the numerous open-ended questions within the tasks themselves, as they were designed to activate students' (a) constant reflection (i.e. assessment) on what they have learnt and (b) constant urge for the exploration of what is yet to be learnt.

Other than these, 'Dialogue' is identified and a 'Checklist' is recommended as assessment tools in this module as well.

(i) Dialogue in classroom, as assessments for and as learning

> "Classroom assessment, both assessment for and as learning, relies on dialogue between the child and the teacher." (Callingham, 2010)

In a lesson with open-ended tasks, exploratory dialogue is expected. Such dialogue, whether student to student or student to teacher, could be developed into conceptual understanding. Dialogue generated in this module helps a teacher identify the student's level of understanding and his/her 'readiness to learn'. Responses from students, both expected and unexpected, can become keys for the teacher to develop strategies towards higher levels of understanding. Therefore, it is an assessment *for* learning.

Follow-up questions can always emerge from a teacher in an exploratory dialogue. The teacher raises those questions as immediate feedback to hint the student to move on with his/her enquiry, towards an extension. This is a process of learning, and so the Q&A dialogue serves as an assessment *as* learning.

(ii) Self-assessing checklist

After Task 3, "Let students derive the area formula for a rectangle", a self-assessing checklist is recommended (see the Resource page). It consists of 4 parts: ❶ Conceptual understanding of the task, ❷ Procedural knowledge and skills, ❸ Problem-solving skills & strategies and ❹ Communication skills. These correspond to the learning targets. The 4 parts above are expressed in academic terms from the teachers' viewpoint. In the checklist for the children to reflect for themselves, they are simplified for their standpoint, namely: 1. How much did I understand the question(s)? 2. How did I find the answer(s)? 3. Which way(s) did I take to find the answer(s)? and 4. Did I listen and express myself well?

With this checklist, students play an active role in self-assessment, rather than being evaluated by the teacher only. They can also be more aware of what they are expected to learn and improve on. The checklist also helps in the assessment of learning in generic skills, values and attitudes, which may not be commonly done.

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